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Journal of Approximation Theory 122 (2003) 42–61
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

Rates of approximation and ergodic limits of regularized operator families

Carlos Lizama* and Humberto Prado

Departamento de Matemática, Universidad de Santiago de Chile, Casilla 307-Correo 2, Santiago, Chile

Received 18 April 2002; accepted 7 February 2003

Communicated by Paul Nevai

Abstract

We state mean ergodic theorems with rates of approximation for a new class of operator families. Our result provides a unified approach to convergence rates for many particular strongly continuous solution families associated to linear evolution equations such as the abstract Cauchy problem of the first and second order, and integral Volterra equations of convolution type. We discuss in particular, applications to α-times integrated cosine families, k-convoluted semigroups and integral resolvents.

Keywords: Strong ergodicity; Uniform ergodicity; Rates of approximation; k-Convoluted semigroup; Integrated sine family; Integral resolvent

1. Introduction

The semigroup approach to the study of differential equations has motivated versions of the classical ergodic theorems for families of operators such as cosine functions, resolvent families, r-times integrated semigroups, and more generally for r-times integrated solutions families [4,11,12,23]. Our aim is to develop the ergodic theorems with rates of approximation for (a,k)-regularized families, a notion which includes that of r-times integrated solution family as well as k-convoluted semigroups, r-times integrated cosine families and integral resolvents among others

*Corresponding author.
E-mail addresses: clizama@usach.cl (C. Lizama), hprado@usach.cl (H. Prado).

0021-9045/03/$ - see front matter
doi:10.1016/S0021-9045(03)00040-6

for which, to our knowledge, rates of approximation and ergodic limits have not been previously studied.

The results we present in this article are pretty much inspired by the works of Shaw [18–20,22]; in particular, paper [23] and the references therein, also by the same author. In [23], Shaw uses the notion of ergodic nets to deduce the strong ergodic and uniform ergodic theorems with rates for r-times integrated solution families associated to the linear Volterra equation of convolution type $u(t) = f(t) + A(a * u)(t)$, $t > 0$, obtaining convergence rates of ergodic limits and approximate solutions of the equation $Ax = y$.

We assume that X is a complex Banach space and let A be a closed linear unbounded operator with domain $D(A)$ defined on X. Consider an arbitrary strongly continuous function $R : \mathbb{R}_+ \to \mathscr{B}(X)$. Suppose that there exist constants $M, \omega \geqslant 0$ such that

$$||R(t)|| \leqslant Me^{\omega t}, \quad t \geqslant 0. \tag{1}$$

Then in such cases, the Laplace transform

$$\hat{R}(\lambda)x := \int_0^\infty e^{-\lambda t} R(t)x\, dt := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} R(t)x\, dt$$

exists for all $\lambda \in \mathbb{C}$ with $Re\, \lambda > \omega$ and all $x \in X$ and defines a bounded operator $\hat{R}(\lambda)$ on X.

Definition 1. *Let $k, a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be Laplace transformable functions and assume there is an $\omega \in \mathbb{R}$ such that $\hat{a}(\lambda) \neq 0$ and $\hat{k}(\lambda) \neq 0$ for all $\lambda > \omega$. We call A the generator of an (a,k)-regularized family if there exists a strongly continuous function $R : \mathbb{R}_+ \to \mathscr{B}(X)$ such that* (1) *holds, $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\lambda > \omega$ and*

$$\hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1} = \int_0^\infty e^{-\lambda s} R(s)\, ds \tag{2}$$

for all $\lambda > \omega$. In this case, R is called the (a,k)-regularized family generated by A.

In this paper, we study the behavior, as $t \to \infty$, of the following family of bounded and linear operators:

$$A_t x := \frac{1}{(k * a)(t)} \int_0^t a(t-s) R(s)x\, ds; \quad x \in X, \ \ t > 0. \tag{3}$$

The family (3) corresponds to the generalized means of the (a,k)-regularized family $\{R(t)\}_{t \geqslant 0}$. In the following examples, we show that the (a,k) regularized resolvent family $\{R(t)\}_{t \geqslant 0}$ is well known for particular choices of the pair (a,k).

Example 1. Suppose that $a(t) = 1$ and $k(t) = 1$. Then, $\{R(t)\}_{t \geqslant 0}$ corresponds to a C_0 *semigroup* generated by A (see [8]) and (3) is the Cesàro mean

$$A_t x = \frac{1}{t} \int_0^t T(s)x\, ds; \quad x \in X, \ \ t > 0. \tag{4}$$

There is an extensive literature about the asymptotic behavior of (4) and its consequences; see e.g. [1,3] and the references therein.

Example 2. If $a(t) = 1$ and $k(t) = \frac{t^n}{n!}$, $n \in \mathbb{N}_0$. Then $\{R(t)\}_{t \geqslant 0}$ corresponds to an *n-times integrated semigroup* (see [10]) and (3) is the average:

$$A_t x = \frac{(n+1)!}{t^{n+1}} \int_0^t R(s)x\, ds; \quad x \in X, \ t > 0. \tag{5}$$

Ergodic Theorems for n-times integrated semigroups have been studied in e.g. [21].

Example 3. If $k(t) = 1$ and a is arbitrary, then $\{R(t)\}_{t \geqslant 0}$ corresponds to a *resolvent family* (see [17]) and the Cèsaro type means are

$$A_t x = \frac{1}{(1 * a)(t)} \int_0^t a(t-s)R(s)x\, ds; \quad x \in X, \ t > 0, \tag{6}$$

whose rates of approximation and ergodic theorems are studied in [4,12].

Example 4. Let $a(t) = 1$ and $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, $\alpha > -1$. Then

$$A_t x = \frac{\Gamma(\alpha+2)}{t^{\alpha+1}} \int_0^t R(s)x\, ds; \quad x \in X, \ t > 0 \tag{7}$$

is a type of means whose convergence rates and ergodic limits have been previously considered by Shaw in [20, Theorem 5], and also ergodic properties, in case of $\alpha = n$, in [19].

Example 5. Suppose that $k(t) = \frac{t^r}{\Gamma(r+1)}$, $r > 0$ and a is arbitrary. Then $\{R(t)\}_{t \geqslant 0}$ corresponds to an *r-times integrated resolvent family*. The means are

$$A_t x = \frac{\Gamma(r+1)}{(t^r * a)(t)} \int_0^t a(t-s)R(s)x\, ds; \quad x \in X, \ t > 0. \tag{8}$$

The above-cited examples show us that the choices of the pair (a, k) classifies different families of strongly continuous operators in $\mathbf{B}(X)$; this fact can be used not only to unify certain earlier results, but also to strengthen and extend these results and to obtain new results. For instance, consider the following examples:

Example 6. Let $a(t) = t$ and $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$. Then, $R(t)$ is the *α-times integrated cosine family* and the means are

$$A_t x = \frac{\Gamma(\alpha+3)}{t^{\alpha+2}} \int_0^t (t-s)R(s)x\, ds; \quad x \in X, \ t > 0. \tag{9}$$

The particular case $\alpha = 1$ was considered before by Cioranescu [5], in order to characterize the infinitesimal generator of a strongly continuous sine function. However, ergodic properties were not investigated.

Example 7. Let $a(t) \equiv 1$ and k is arbitrary. Then, $R(t)$ is a *k-convoluted semigroup*, see [6,7]; those ergodic properties, to our knowledge, have not been previously studied. In this case, the averages (3) are given by

$$A_t x = \frac{1}{(1 * k)(t)} \int_0^t R(s)x\, ds; \quad x \in X, \ \ t > 0. \tag{10}$$

Example 8. Let $a(t) = k(t)$ be arbitrary. In this case, $R(t)$ is the *integral resolvent* and our definition gives us a slight modification of the concept introduced in [17, Definition 1.6].

The ergodic theorems developed in this article include all the above-cited examples with their corresponding means $(A_t)_{t>0}$. Furthermore, we give a new formulation of the previous results obtained by Shaw [23] to study the convergence of ergodic limits with rates for r-times integrated solution families; see Remark 3 at the end of Section 2.

Henceforth, we assume the following conditions, which will be called *condition* (C): Let $a(t)$ be positive and $k(t)$ nondecreasing and positive as well, satisfying

$$\text{(C1)} \quad \lim_{t\to\infty} \frac{k(t)}{(k * a)(t)} = 0,$$

$$\text{(C2)} \quad \sup_{t>0} \frac{k(t)(1 * a)(t)}{(k * a)(t)} < \infty,$$

$$\text{(C3)} \quad \lim_{t\to\infty} \frac{(a * a * k)(t)}{(a * k)(t)} = \infty.$$

Remark 1. The fact that $k(t)$ is nondecreasing implies that $(k * a)(t) \leqslant (1 * a)(t)k(t)$ for all $t \geqslant 0$. Hence, $\frac{1}{(1*a)(t)} \leqslant \frac{k(t)}{(k*a)(t)}$ and then it follows that $(1 * a)(t) \to \infty$ as $t \to \infty$, by (C1). These remarks will be needed in the following sections.

2. Ergodic theorems with rates of approximation

We first recall the following definition due to Shaw from [18], and this will be the main tool in the development of this article.

Definition 2. Suppose A is a closed linear operator with domain $D(A)$ defined on a complex Banach space X, and let A_α and B_α be two nets of bounded operators on X satisfying

(E1) $\sup_\alpha \|A_\alpha\| \leqslant M$,
(E2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subseteq AB_\alpha = I - A_\alpha$ for all α,
(E3) $R(A_\alpha) \subset D(A)$ for all α and $\|AA_\alpha\| = O(e(\alpha))$ with $e(\alpha) \to 0$,

(E4) $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$ with $|\varphi(\alpha)| \to \infty$,

(E5) $||A_\alpha y|| = O(f(\alpha))$ (resp. $o(f(\alpha))$) implies $||B_\alpha y|| = O(f(\alpha)/e(\alpha))$ (resp. $o(f(\alpha)/e(\alpha))$, where e and f are such that $0 < e(\alpha) \leqslant f(\alpha)$ and $f(\alpha) \to 0$.

The net $(A_\alpha)_\alpha$ is called an A-ergodic net and $(B_\alpha)_\alpha$ is called the companion net.

Remark 2. The functions e and f act as estimators of the convergence rates of $(A_\alpha x)_\alpha$ and $(B_\alpha y)_\alpha$, which, in practical applications, approximate the ergodic limit and the solution of the equation $Ax = y$, respectively.

Let $x \in X$ be given and define

$$B_t x = \frac{-1}{(k * a)(t)} \int_0^t a(t-s) \int_0^s a(s-r) R(r) x \, dr \, ds \tag{11}$$

for all $t > 0$. We will prove that $(A_t)_{t>0}$ defined by (3) is A-ergodic and that $(B_t)_{t>0}$ is the companion net.

Under the hypothesis that $k(t)$ is continuous, it was shown by Lizama [13] (Proposition 3.1) that Definition 1 is equivalent to the following three properties:

(R1) $R(0) = k(0)I$;

(R2) $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t > 0$;

(R3)

$$R(t)x = k(t)x + \int_0^t a(t-s) A R(s) x \, ds \tag{12}$$

for all $x \in D(A)$ and $t \geqslant 0$.

In the sequel, we will always assume that $k(t)$ is continuous.

Let P and B_1 be the operators defined by

$$Px := \lim_{t \to \infty} A_t x; \quad B_1 y := \lim_{t \to \infty} B_t y$$

on their maximal domains, respectively. The family $\{A_t\}$ is said to be strongly (resp. uniformly) ergodic if $D(P) = X$ and $A_t x \to Px$ as $t \to \infty$ for all $x \in X$ (resp. $||A_t - P|| \to 0$ as $t \to \infty$.)

The rates of convergence of ergodic limits are characterized by means of the *K-functional* and the *relative completion*, which we recall as follows:

Let X be a Banach space with norm $||\cdot||_X$ and Y a submanifold with seminorm $||\cdot||_Y$. The *K-functional* is defined by

$$K(t,x) := K(t,x,X,Y,||\cdot||_Y) = \inf_{y \in Y} \{||x-y||_X + t||y||_Y\}.$$

If Y is a Banach space with norm $||\cdot||_Y$, the completion of Y relative to X is defined as

$$\tilde{Y}^X := \Big\{ x \in X: \text{ there exists } \{x_m\} \subset Y \text{ such that } \lim_{m \to \infty} ||x_m - x||_X = 0 \text{ and } \sup_m ||x_m|| < \infty \Big\}.$$

Since the operator $B_1 : D(B_1) \subset \overline{Ran(A)} \to \overline{Ran(A)}$ is closed, its domain $D(B_1)$ is a Banach space with respect to the graph norm. Let $B_0 : D(B_0) \subset Ker(A) \oplus \overline{Ran(A)} \to Ker(A) \oplus \overline{Ran(A)}$ be the operator $B_0(x+y) := B_1 y$; Hence, its domain $D(B_0)$ is also a Banach space with respect to the graph norm, and $[\widetilde{D(B_0)}]_{Ker(A)\oplus\overline{Ran(A)}} = Ker(A) \oplus [\widetilde{D(B_1)}]_{\overline{Ran(A)}}$.

Our first main result in this paper is the following:

Theorem 1 (Strong Ergodic Theorem with rates). *Let A be the generator of an (a,k)-regularized family $\{R(t)\}_{t\geqslant 0}$ such that*

$$||R(t)|| \leqslant Mk(t) \quad \textit{for all } t \geqslant 0.$$

Suppose that $(A_t)_{t>0}$ and $(B_t)_{t>0}$ are defined by (3) *and* (11), *respectively. Then, under condition* (C) *the following are satisfied*:

(i) *The mapping $Px := \lim_{t\to\infty} A_t x$ is a bounded linear projection with $Ran(P) = Ker(A)$, $Ker(P) = \overline{Ran(A)}$, and*

$$D(P) = Ker(A) \oplus \overline{Ran(A)} = \{x \in X; \{A_t\} \textit{ has a weak cluster point}\}.$$

For $0 < \beta \leqslant 1$ and $x \in Ker(A) \oplus \overline{Ran(A)}$, we have

$$||A_t x - Px|| = O\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right)$$

$$\Leftrightarrow K\left(\frac{k(t)}{(a*k)(t)}, x, Ker(A) \oplus \overline{Ran(A)}, D(B_0), ||\cdot||_{B_0}\right) = O\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right)$$

$$\Leftrightarrow x \in [\widetilde{D(B_0)}]_{Ker(A)\oplus\overline{Ran(A)}} \quad (\textit{in case } \beta = 1).$$

(ii) *$B_1 y = -\lim_{t\to\infty} \frac{1}{(a*k)(t)}(a*a*R)(t)y$ is the inverse operator A_1^{-1} of the restriction $A_1 := A|_{\overline{Ran(A)}}$ of A to $\overline{Ran(A)}$; it has range $Ran(B_1) = D(A) \cap \overline{Ran(A)}$ and domain $D(B_1) = A(D(A) \cap \overline{Ran(A)})$. Moreover, for each $y \in D(B_1)$, $B_1 y$ is the unique solution of the functional equation $Ax = y$ in $\overline{Ran(A)}$, and we have for $0 < \beta \leqslant 1$,*

$$\left|\left|\frac{1}{(a*k)(t)}(a*a*R)(t)y + A_1^{-1}y\right|\right| = O\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right)$$

$$\Leftrightarrow K\left(\frac{k(t)}{(a*k)(t)}, B_1 y, \overline{Ran(A)}, D(B_1), ||\cdot||_{B_1}\right) = O\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right)$$

$$\Leftrightarrow y \in A(D(A) \cap [\widetilde{D(B_1)}]_{\overline{Ran(A)}}) \quad (\textit{in case } \beta = 1).$$

(iii) *$Ran(A)$ is not closed if and only if for every* (*some*) *$0 < \beta < 1$ there exists an element $y_\beta \in \overline{Ran(A)}$ such that*

$$||A_t y_\beta|| = O\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right) \quad \left(||A_t y_\beta|| \neq o\left(\left[\frac{k(t)}{(a*k)(t)}\right]^\beta\right)\right).$$

Proof. We will prove that the net $(A_t)_{t>0}$ is A-ergodic and $(B_t)_{t>0}$ is the companion net, with

$$e(t) = \frac{k(t)}{(a*k)(t)} \quad \text{and} \quad f(t) = e(t)^{\beta}, \qquad 0<\beta\leqslant 1,$$

for all $t>0$, according to Definition 2.

(E1) Since $||R(s)||\leqslant Mk(s)$ for all $s\geqslant 0$ and $a(t)$, $k(t)$ are nonnegative functions, we obtain

$$\begin{aligned}||A_t x|| &= \left|\left|\frac{1}{(k*a)(t)}\int_0^t a(t-s)R(s)x\,ds\right|\right| \\ &\leqslant \frac{M}{(k*a)(t)}\int_0^t a(t-s)k(s)\,ds||x|| \\ &= M||x||\end{aligned}$$

for all $x\in X$ and $t>0$.

(E2) For $x\in X$ and $t\geqslant 0$, define $S(t)x = \int_0^t a(t-s)R(s)x\,ds$. Then, by (R2) we obtain for all $x\in D(A)$ that $S(t)x\in D(A)$ and $AS(t)x = S(t)Ax$. Then, $B_t x\in D(A)$ and $AB_t x = B_t Ax$ for all $x\in D(A)$ and all $t\geqslant 0$. Since $Ran(S(t))\subset D(A)$ (see [13, Lemma 2.2]) and A is closed we get $B_t x\in D(A)$ for all $x\in X$, that is $Ran(B_t)\subset D(A)$ and $B_t A\subseteq AB_t$ for all $t\geqslant 0$. On the other hand,

$$\begin{aligned}AB_t x &= \frac{-1}{(k*a)(t)}\int_0^t a(t-s)\left[\int_0^s a(s-r)AR(r)x\,dr\right]ds \\ &= \frac{-1}{(k*a)(t)}\int_0^t a(t-s)[R(s)x-k(s)x]\,ds \\ &= -A_t x + x.\end{aligned}$$

Hence, $AB_t = I - A_t$ for all $t\geqslant 0$.

(E3) For all $t\geqslant 0$, define the family of operators $S(t)$ as in (E2) above. Since $Ran(S(t))\subset D(A)$ it follows that $R(A_t)\subset D(A)$ for all $t\geqslant 0$. Then using (C1) we obtain

$$\begin{aligned}||AA_t|| &= \left|\left|\frac{1}{(k*a)(t)}\int_0^t a(t-s)AR(s)x\,ds\right|\right| \\ &= \left|\left|\frac{R(t)x-k(t)x}{(k*a)(t)}\right|\right| \\ &\leqslant (M+1)\frac{k(t)}{(k*a)(t)}||x||\to 0^+ \quad \text{as } t\to\infty.\end{aligned}$$

Hence, $||AA_t|| = O(\frac{k(t)}{(k*a)(t)}) = O(e(t))$.

(E4) Let $x^*\in N(A^*)$. Then

$$\begin{aligned}\langle x, R^*(t)x^*\rangle &= \left\langle x, k(t)x^* + \int_0^t a(t-s)R^*(s)A^*x^*\,ds\right\rangle \\ &= \langle x, k(t)x^*\rangle\end{aligned}$$

for all $x \in X$. Thus, $R^*(t)x^* = k(t)x^*$ and hence, if we define $\varphi(t) = \frac{-1}{(k*a)(t)} \int_0^t a(t-s)(a*k)(s)\,ds$ we obtain

$$\begin{aligned} B_t^* x^* &= \frac{-1}{(k*a)(t)} \int_0^t a(t-s) \int_0^s a(s-r) R^*(r) x^* \, dr \, ds \\ &= \frac{-1}{(k*a)(t)} \left[\int_0^t a(t-s) \int_0^s a(s-r) k(r) \, dr \, ds \right] x^* \\ &= \frac{-1}{(k*a)(t)} \left[\int_0^t a(t-s)(a*k)(s) \, ds \right] x^* \\ &= \varphi(t) x^*, \end{aligned}$$

where, in view of (C3),

$$|\varphi(t)| = \frac{(a*a*k)(t)}{(k*a)(t)} \to \infty \quad \text{as } t \to \infty.$$

(E5) Let $y \in X$ be such that

$$||A_t y|| \leqslant M(y) \left[\frac{k(t)}{(k*a)(t)} \right]^\beta, \quad 0 < \beta \leqslant 1,$$

and define $f(t) = e(t)^\beta$. Since $\frac{k(t)}{(k*a)(t)} \to 0$ as $t \to \infty$ implying that $(\frac{k(t)}{(k*a)(t)})^{1-\beta} \leqslant 1$ for large t and $0 < \beta \leqslant 1$. Thus, $0 < e(t) \leqslant f(t)$ as $t \to \infty$, and

$$\begin{aligned} ||B_t y|| &= \left\| \frac{1}{(k*a)(t)} \int_0^t a(t-s)(k*a)(s) \right. \\ &\quad \left. \times \left(\frac{1}{(k*a)(s)} \int_0^s a(s-r) R(r) y \, dr \right) ds \right\| \\ &= \left\| \frac{1}{(k*a)(t)} \int_0^t a(t-s)(k*a)(s) A_s y \, ds \right\| \\ &\leqslant \frac{1}{(k*a)(t)} \left(\int_0^t a(t-s) k(s) e(s)^{\beta-1} \, ds \right) M(y) \\ &= \frac{1}{(k*a)(t)} \left(\int_0^t a(t-s) k(s) e(s)^{\beta-1} e(t)^{1-\beta} \, ds \right) e(t)^{\beta-1} M(y) \\ &= \frac{1}{(k*a)(t)} \left(\int_0^t a(t-s) k(s) \left(\frac{k(s)}{(a*k)(s)} \right)^{\beta-1} \left(\frac{(a*k)(t)}{k(t)} \right)^{\beta-1} ds \right) \\ &\quad \times e(t)^{\beta-1} M(y) \\ &= \frac{1}{(k*a)(t)} \left(\int_0^t a(t-s) k(s) \left(\frac{k(s)}{k(t)} \right)^\beta \left(\frac{(a*k)(s)}{(a*k)(t)} \right)^{1-\beta} \frac{k(t)}{k(s)} \, ds \right) \\ &\quad \times e(t)^{\beta-1} M(y) \end{aligned}$$

for all $t>0$. Note that $t \to (a*k)(t)$ and $t \to k(t)$ are nondecreasing. Hence, for $0<\beta\leqslant 1$ we have $(\frac{(a*k)(s)}{(a*k)(t)})^{1-\beta}\leqslant 1$ and $(\frac{k(s)}{k(t)})^{\beta}\leqslant 1$ whenever $0<s<t$. In view of (C2) we get that the last equation is bounded by $e(t)^{\beta-1}M(y)$. Hence, $||B_t y|| = O(e(t)^{\beta-1})$.

Finally, applying the abstract Strong Ergodic Theorem from [18] and Theorems 1 and 2 from [22] concerned with optimal convergence and non-optimal convergence rates of ergodic limits, we obtain the assertions of the theorem. □

A Banach space X is called a *Grothendieck space* if every weakly* convergent sequence in X^* is weakly convergent and is to said to have the *Dunford–Pettis property* if every weakly compact operator from X to any Banach space maps weakly compact sets into norm compact sets. The spaces L^∞, H^∞, and $B(S,\Sigma)$ are particular examples of Grothendieck spaces with the Dunford–Pettis property.

Let A be an operator on X, and let Y be a closed subspace of X. The *part* of A in Y is the operator A_Y on Y defined by

$$D(A_Y) := \{y \in D(A)\cap Y\colon\ Ay \in Y\},$$

$$A_Y y := Ay.$$

Theorem 2 (Uniform Ergodic Theorem). *Let A be the generator of an (a,k)-regularized family $\{R(t)\}_{t\geqslant 0}$ such that*

$$||R(t)|| \leqslant Mk(t) \quad \textit{for all } t\geqslant 0,$$

then

(a) *Under condition* (C1) *the following assertions are equivalent*:

(i) $\{A_t\}$ *is uniformly ergodic*;

(ii) $||B_t \mid Ran(A)|| = O(1)$;

(iii) B_1 *is bounded and* $||B_t \mid Ran(A) - B_1^{-1}|| \to 0$ *as* $t\to\infty$;

(iv) $Ran(A)$ *is closed*;

(v) $Ran(A^2)$ *is closed*;

(vi) $X = Ker(A) \oplus Ran(A)$.

Moreover, the convergence of the limits has order $O(\frac{k(t)}{(a*k)(t)})$.

(b) *Assume condition* (C) *and if $\overline{D(A)}$ is a Grothendieck space with the Dunford–Pettis property, and $\overline{D(A)}\subset D(P)$, then* $||A_t|_{\overline{D(A)}} - P|_{\overline{D(A)}}|| = O(\frac{k(t)}{(a*k)(t)})$.

Proof. Under condition (C1), we have that (E1)–(E3) are satisfied. Thus assertions in (a) follow from the Uniform Ergodic Theorem stated in [19, Theorem 1].

To prove (b), it suffices to show that the part of A in $\overline{D(A)}$ has dense domain in $\overline{D(A)}$ and hence, the assertion follows from Shaw [23, Theorem 2.3]. In fact, let $x\in D(A)$, then by (R2) we have that $A_t x \in D(A)$ for all $t>0$. Moreover, by (R3),

$$AA_t x = \frac{1}{(a*k)(t)} R(t)x - \frac{k(t)}{(a*k)(t)} x \in D(A).$$

Hence, $A_t x \in D(A_{\overline{D(A)}})$. On the other hand, note that for $x \in D(A)$,

$$\begin{aligned}\left\|\frac{R(t)}{k(t)}x - x\right\| &\leqslant \frac{1}{k(t)}\int_0^t a(t-s)\|R(s)Ax\|\,ds\\ &\leqslant M\int_0^t a(t-s)\frac{k(s)}{k(t)}\,ds\|Ax\|\\ &\leqslant M\int_0^t a(s)\,ds\|Ax\|,\end{aligned}$$

since $k(s)/k(t) \leqslant 1$ whenever $s \leqslant t$. Thus, for all $x \in D(A)$, $\frac{R(t)}{k(t)}x \to x$ as $t \to 0^+$. Hence, given $\varepsilon > 0$ there is $t_0 > 0$ such that

$$\begin{aligned}\|A_t x - x\| &= \left\|\frac{1}{(a*k)(t)}(a*R)(t)x - x\right\|\\ &= \left\|\frac{1}{(a*k)(t)}[(a*R)(t)x - (a*k)(t)x]\right\|\\ &= \frac{1}{(a*k)(t)}\left\|\int_0^t a(t-s)[R(s)x - k(s)x]\,ds\right\|\\ &\leqslant \frac{1}{(a*k)(t)}\int_0^t a(t-s)k(s)\left\|\frac{R(s)}{k(s)}x - x\right\|\,ds < \varepsilon,\end{aligned}$$

whenever $t < t_0$. $\square$

Proposition 1. *Suppose that A is the generator of an (a,k)-regularized family $\{R(t)\}_{t\geqslant 0}$, where $a(t)$ and $k(t)$ are Laplace transformable and $\hat{a}(\lambda) \neq 0$, $\hat{k}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > 0$. Assume also that condition* (C1) *is satisfied and*

$$\|R(t)\| \leqslant Mk(t) \quad \text{for all } t \geqslant 0.$$

If $\hat{a}(\lambda)$ has a pole in $\mathbb{C}_+$, then $X = Ker(A) \oplus Ran(A)$, and all the assertions in part (a) *of the uniform ergodic theorem are always true.*

Proof. Let $\lambda_0 \in \mathbb{C}_+$ be a pole of $\hat{a}(\lambda)$ of order n. Then there is an $\epsilon > 0$ such that $V := \{z \in \mathbb{C}\colon 0 < |z| < \epsilon\} \subset \rho(A)$. For all λ near to λ_0, we have

$$\begin{aligned}\left\|\left(\frac{1}{\hat{a}(\lambda)} - A\right)^{-1}\right\| &= \|\hat{a}(\lambda)(I - \hat{a}(\lambda)A)^{-1}\|\\ &= \left\|\frac{\hat{a}(\lambda)}{\hat{k}(\lambda)}\hat{R}(\lambda)\right\|\\ &\leqslant M|\hat{a}(\lambda)|.\end{aligned}$$

Hence, for all $z \in V$,

$$\|(z - A)^{-1}\| \leqslant M/|z|.$$

Thus, zero is at most a pole of order 1 of $(z-A)^{-1}$. Now the claim follows from [24, Chapter VIII.8] and Theorem 2. $\square$

Remark 3. We point out that we can take control of the approximation rates in Theorem 2 above by taking m-times the convolution product of $a(t)$ with itself. Towards this end, let $m \in \mathbb{N}$ be given and define

$$a_1(t) := a(t), \quad a_m(t) := (a * a * \cdots * a)(t) \quad \text{for } m \geqslant 2.$$

If we replace in Eq. (3) the function $a(t)$ by $a_m(t)$ then, whenever $R(t)$ is an (a,k)-regularized family, we obtain the means

$$A_t^m x = \frac{1}{(k * a_m)(t)} \int_0^t a_m(t-s) R(s) x \, ds \quad t > 0, \ x \in X$$

and

$$B_t^m x = \frac{-1}{(k * a_m)(t)} \int_0^t a_{m+1}(t-s) R(s) x \, ds \quad t > 0, \ x \in X.$$

In particular, for $k(t) = \frac{t^r}{\Gamma(r+1)}$ we recover the approximation processes $Q_m(t)$ defined by Shaw in [23, p. 364] (see also [11]), and the condition (C) can be stated as follows:

(C_m) For a fixed $m \in \mathbb{N}$, suppose that $a(t)$ is positive and $k(t)$ is nondecreasing and positive, satisfying

$$\lim_{t \to \infty} \frac{k(t)}{(k * a_m)(t)} = 0,$$

$$\sup_{t > 0} \frac{k(t)(1 * a_m)(t)}{(k * a_m)(t)} < \infty,$$

and

$$\lim_{t \to \infty} \frac{(a_{2m} * k)(t)}{(a_m * k)(t)} = \infty.$$

It is not difficult to see that if $\{R(t)\}_{t \geqslant 0}$ is an (a,k)-regularized family such that

$$||R(t)|| \leqslant M k(t), \quad t \geqslant 0,$$

then under conditions (C_m) the hypothesis (E1)–(E5) are satisfied. Thus, all the conclusions in this section remain true with the modifications indicated above.

An Example. Let $X := C[0,1]$ and let $A : D(A) \to X$ be defined by $Af = f'$ where $D(A) := \{f \in C^1[0,1]: f(0) = f(1)\}$. Let $f \in X$, then

$$f(x) = \int_0^1 f(s) \, ds + A\left[\int_0^x \left(f(s) - \int_0^1 f(t) \, dt \right) ds\right].$$

Moreover, for $f \in Ker(A) \cap Ran(A)$, $Af = 0$ and $f = Ag = g'$, $g \in D(A)$. But $f' = 0$ and $g' = f$ implies $f(x) = c$ and $g'(x) = c$. We conclude that $g(x) = cx$. On the other hand, $g \in D(A)$ implies $0 = g(0) = g(1) = c$. Therefore, $c = 0$, showing that $f(x) = 0$. Hence, we obtain that $Ker(A) \cap Ran(A) = \{0\}$ and

$$X = Ker(A) \oplus Ran(A).$$

In particular, we have

$$P(f)(t) = \int_0^1 f(s)\, ds \quad \text{for all } t \in [0, 1].$$

It is not difficult to see that A is the generator of a once-integrated semi-group $\{S(t)\}_{t \geqslant 0}$ with $||S(t)|| \leqslant Mt$ for all $t \geqslant 0$. Let $a(t)$ be real and positive. Also suppose that $a(t)$ is a completely positive kernel (see [17] for definitions). Then, by [13, Theorem 3.7], A is the generator of a (a,k)-regularized family $\{R(t)\}_{t \geqslant 0}$ where

$$k(t) = (1 * a)(t).$$

Clearly, $k(t)$ is positive and nondecreasing. Hence, assuming that

$$\lim_{t \to \infty} \frac{(1 * a)(t)}{(1 * a * a)(t)} = 0,$$

we have by (a) of Theorem 2,

$$\lim_{t \to \infty} \frac{1}{(1 * a * a)(t)} \int_0^t a(t-s)(R(s)f)(\tau)\, ds = \int_0^1 f(s)\, ds, \quad \tau \in [0, 1].$$

3. Abelian ergodic theorems with rates of approximation

Assume that $a(t)$ and $k(t)$ are of subexponential growth, i.e. $\int_0^\infty e^{-\varepsilon t} a(t)\, dt < \infty$ for all $\varepsilon > 0$. Then $\hat{a}(\lambda)$, the Laplace transform of $a(t)$ (resp. $k(t)$) exists for all $\lambda > 0$. In the sequel, we suppose that $\hat{a}(\lambda) \neq 0$ and $\hat{k}(\lambda) \neq 0$ for all $\lambda > 0$.

Assume $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $Re\, \lambda > 0$ and define the nets

$$A_\lambda := \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} - A \right)^{-1} \quad \text{for all } \lambda > 0$$

and

$$B_\lambda := -\left(\frac{1}{\hat{a}(\lambda)} - A \right)^{-1} \quad \text{for all } \lambda > 0.$$

Note that under condition (C1), we have $\lim_{\lambda \to 0^+} \frac{1}{\hat{a}(\lambda)} = 0$, since $\lim_{\lambda \to 0} \hat{a}(\lambda) = \lim_{t \to \infty} (1 * a)(t) = \infty$ by Remark 1. Thus, $0 \in \overline{\rho(A)}$. Also by the positivity of $k(t)$, $\hat{k}(0) \neq 0$ and hence $\sup_{0 < \lambda \leqslant 1} \frac{1}{\hat{k}(\lambda)} < \infty$.

Proposition 2. *Suppose A is the generator of an (a,k)-regularized family $\{R(t)\}_{t\geqslant 0}$ such that*

$$||R(t)||\leqslant Mk(t) \quad \textit{for all } t\geqslant 0.$$

Then, under (C1), *the net $(A_\lambda)_{\lambda>0}$ is A-ergodic and $(B_\lambda)_{\lambda>0}$ is the companion net, with*

$$e(\lambda)=\frac{1}{\hat{a}(\lambda)} \quad \textit{and} \quad f(\lambda)=e(\lambda)^\beta, \qquad 0<\beta\leqslant 1,$$

for all $\lambda>0$.

Proof. To show (E1) note that $||R(t)||\leqslant Mk(t)$. Then $||\hat{R}(\lambda)||\leqslant M\hat{k}(\lambda)$ for $\lambda>0$. Moreover,

$$(I-\hat{a}(\lambda)A)\hat{R}(\lambda)=\hat{k}(\lambda)I, \quad \lambda>0,$$

by Definition 1. Hence,

$$A_\lambda=\frac{\hat{R}(\lambda)}{\hat{k}(\lambda)},$$

thus $||A_\lambda||=\frac{||\hat{R}(\lambda)||}{\hat{k}(\lambda)}\leqslant M$, for $\lambda>0$. On the other hand, $B_\lambda=-\hat{a}(\lambda)A_\lambda$. Hence,

$$\begin{aligned}B_\lambda A+A_\lambda &= -\hat{a}(\lambda)A_\lambda A+A_\lambda\\ &=A_\lambda(I-\hat{a}(\lambda)A)\\ &=\frac{1}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)}-A\right)^{-1}(I-\hat{a}(\lambda)A)\\ &=I\end{aligned}$$

on $D(A)$. Since A commutes with B_λ and A_λ, it follows that $B_\lambda A\subseteq I-A_\lambda$ for $\lambda>0$, and (E2) follows.

To prove (E3), note that by Definition 1

$$AA_\lambda=\frac{1}{\hat{a}(\lambda)}(A_\lambda-I).$$

Then $||AA_\lambda||\leqslant(M+1)\frac{1}{\hat{a}(\lambda)}=(M+1)e(\lambda)$, where $e(\lambda)=\frac{1}{\hat{a}(\lambda)}\to 0$ as $\lambda\to 0$ by (C1) and Remark 1.

To show (E4), note that

$$\begin{aligned}B_\lambda-\hat{a}(\lambda)AB_\lambda &=(I-\hat{a}(\lambda)A)B_\lambda\\ &= -\hat{a}(\lambda)\left(\frac{1}{\hat{a}(\lambda)}-A\right)\left(\frac{1}{\hat{a}(\lambda)}-A\right)^{-1}\\ &= -\hat{a}(\lambda)I.\end{aligned}$$

Thus, $B_\lambda = -\hat{a}(\lambda)(I - AB_\lambda)$. Now for $x^* \in Ran(A)^\perp$ and $x \in X$ it follows that

$$\begin{aligned}\langle x, B_\lambda^* x^* \rangle &= \langle B_\lambda x, x^* \rangle \\ &= -\hat{a}(\lambda)\langle (I - AB_\lambda)x, x^* \rangle \\ &= -\hat{a}(\lambda)\langle x, x^* \rangle + \hat{a}(\lambda)\langle AB_\lambda x, x^* \rangle \\ &= -\hat{a}(\lambda)\langle x, x^* \rangle.\end{aligned}$$

Hence, $B_\lambda^* x^* = -\hat{a}(\lambda)x^*$ for $x^* \in Ran(A)^\perp$ and since $\lim_{\lambda\to 0^+} \hat{a}(\lambda) = \infty$, the proof of (E4) follows.

To show (E5), assume that $||A_\lambda|| \leqslant Mf(\lambda)$, where $f(\lambda) := e(\lambda)^\beta$ $(0 < \beta \leqslant 1)$. Notice that $0 < e(\lambda) \leqslant f(\lambda)$ since $\frac{1}{\hat{a}(\lambda)} \to 0$ as $\lambda \to 0^+$ obtaining that $(\frac{1}{\hat{a}(\lambda)})^{1-\beta} < 1$.

Since $B_\lambda = -\hat{a}(\lambda)A_\lambda$,

$$\begin{aligned}||B_\lambda|| &= \hat{a}(\lambda)||A_\lambda|| \\ &\leqslant M\hat{a}(\lambda)f(\lambda) \\ &= M\frac{f(\lambda)}{e(\lambda)}.\end{aligned}$$

Hence, $||B_\lambda|| = O(\frac{f(\lambda)}{e(\lambda)})$. Now if $||A_\lambda|| = o(f(\lambda))$ as $\lambda \to 0^+$, then

$$\frac{1}{\hat{a}(\lambda)f(\lambda)}||B_\lambda|| = \frac{1}{f(\lambda)}||A_\lambda|| \to 0$$

as $\lambda \to 0^+$. Hence,

$$\begin{aligned}||B_\lambda|| &= o(\hat{a}(\lambda)f(\lambda)) \\ &= o\left(\frac{f(\lambda)}{e(\lambda)}\right)\end{aligned}$$

as $\lambda \to 0^+$. $\square$

Applying Proposition 2 together with Theorems 1 and 2 of [22] and 2.3 of [23], we obtain

Theorem 3 (Strong Ergodic Theorem with rates). *Let A be the generator of an (a, k)-regularized family $\{R(t)\}_{t\geqslant 0}$ such that condition* (C1) *is satisfied and*

$$||R(t)|| \leqslant Mk(t) \quad \textit{for all } t \geqslant 0.$$

Then

(i) *For $0 < \beta \leqslant 1$ and $x \in Ker(A) \oplus \overline{Ran(A)}$, we have*

$$\begin{aligned}&||A_\lambda x - Px|| = O([\hat{a}(\lambda)]^{-\beta}) \\ &\Leftrightarrow K([\hat{a}(\lambda)]^{-1}, x, Ker(A) \oplus \overline{Ran(A)}, D(B_0), ||\cdot||_{B_0}) = O([\hat{a}(\lambda)]^{-\beta}) \\ &\Leftrightarrow x \in [\widetilde{D(B_0)}]_{Ker(A)\oplus\overline{Ran(A)}} \quad (\textit{in case } \beta = 1).\end{aligned}$$

(ii) *For each* $y \in D(B_1)$ *and* $0 < \beta \leqslant 1$, *we have*

$$||A_\lambda y + A_1^{-1}y|| = O([\hat{a}(\lambda)]^{-\beta})$$
$$\Leftrightarrow K([\hat{a}(\lambda)]^{-1}, B_1 y, \overline{Ran(A)}, D(B_1), ||\cdot||_{B_1}) = O([\hat{a}(\lambda)]^{-\beta})$$
$$\Leftrightarrow y \in A(D(A) \cap [\widetilde{D(B_1)}]_{\overline{Ran(A)}}) \quad (\text{in case } \beta = 1).$$

(iii) $Ran(A)$ *is not closed if and only if for every* (*some*) $0 < \beta < 1$ *there exists an element* $y_\beta \in \overline{Ran(A)}$ *such that*

$$||A_\lambda y_\beta|| = O([\hat{a}(\lambda)]^{-\beta}) \quad \text{and} \quad ||A_\lambda y_\beta|| \neq o([\hat{a}(\lambda)]^{-\beta}).$$

Theorem 4 (Uniform Ergodic Theorem). *Let* A *be the generator of an* (a,k)-*regularized family* $\{R(t)\}_{t \geqslant 0}$ *such that condition* (C1) *is satisfied and*

$$||R(t)|| \leqslant Mk(t) \quad \text{for all } t \geqslant 0.$$

Then

(a) *the following assertions are equivalent*
(i) $D(P) = X$ *and* $||A_\lambda - P|| \to 0$ *as* $t \to \infty$;
(ii) $||B_\lambda \mid Ran(A)|| = O(1)$;
(iii) B_1 *is bounded and* $||B_\lambda \mid Ran(A) + A_1^{-1}|| \to 0$ *as* $t \to \infty$;
(iv) $Ran(A)$ *is closed*;
(v) $Ran(A^2)$ *is closed*;
(vi) $X = Ker(A) \oplus Ran(A)$.
Moreover, the convergence of the limits has order $O([\hat{a}(\lambda)]^{-1})$.

(b) *If* $\overline{D(A)}$ *is a Grothendieck space with the Dunford–Pettis property, and if* $\overline{D(A)} \subset D(P)$, *then* $||A_\lambda|_{\overline{D(A)}} - P|_{\overline{D(A)}}|| = O([\hat{a}(\lambda)]^{-1})$.

4. Application to r-times integrated solution families

A family $\{R(t)\}_{t \geqslant 0}$ in $\mathbf{B}(X)$ is called an r-times integrated resolvent family if $\{R(t)\}_{t \geqslant 0}$ satisfies Definition 1 with $k(t) = \frac{t^r}{\Gamma(r+1)}$ (see [2,16] for the case $r = n \in \mathbb{N}$ and [23] for the general case).

The r-times integrated solution families allow one to find and study the behavior of the solution for the following integral equation:

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \geqslant 0, \tag{13}$$

where $f \in L^1(\mathbb{R}_+; X)$. Ergodic properties of r-times integrated resolvent families have been discussed in the paper [23]. In this section, we recover Shaw's results in terms of the following means of Cesàro type (see Remark 3):

$$A_t^m x = \frac{\Gamma(r+1)}{(t^r * a_m)(t)} \int_0^t a_m(t-s)R(s)x\,ds, \quad t > 0, \quad x \in X,$$

with companion net

$$B_t^m x = \frac{-\Gamma(r+1)}{(t^r * a_m)(t)} (a_{m+1} * R)(t)x, \quad t>0, \ x \in X.$$

Observe that condition (C) now takes the form
$(\mathrm{C}_m)'$ $a(t)$ is positive and, for fixed $m \in \mathbb{N}$,

$$\lim_{t\to\infty} \frac{t^r}{(t^{r-2} * a_m)(t)} = 0,$$

$$\lim_{t\to\infty} \frac{t^r(1 * a_m)(t)}{(t^r * a_m)(t)} < \infty,$$

$$\lim_{t\to\infty} \frac{(t^r * a_{2m})(t)}{(t^{r-2} * a_m)(t)} = \infty.$$

Now, applying Theorem 1 we can state the following strong ergodic theorem with rates (see Shaw [23, Theorem 3.2, p. 365]).

Theorem 5. *Let* $\{R(t)\}_{t\geqslant 0}$ *be an r-times integrated solution family for Eq.* (13) *satisfying*

$$||R(t)|| \leqslant M_r t^r \quad \textit{for all } t \geqslant 0$$

and assume condition $(\mathrm{C}_m)'$. *Then the same conclusions of Theorem* 1 *are true, with*

$$Px = \lim_{t\to\infty} A_t^m x$$

for all $x \in D(P) = Ker(A) \oplus \overline{Ran(A)}$, *and*

$$B_1 y = -\lim_{t\to\infty} \frac{\Gamma(r+1)}{(a_m * t^r)(t)} (a_{m+1} * R)(t)y$$

for all $x \in D(B_1) = A(D(A) \cap \overline{Ran(A)})$. *Moreover, the rate of convergence has order* $\frac{t^r}{(a_m * t^r)(t)}$.

Recall that a p-times integrated semigroup ($p \in \mathbb{N}_0$) is called *tempered* if

$$||R(t)|| \leqslant ct^p \quad (t \geqslant 0)$$

for some $c \geqslant 0$. A direct application of the above results is the following:

Proposition 3. *Let* $\{R(t)\}_{t\geqslant 0}$ *be an r-times integrated tempered semigroup with generator A. Then for fixed* $m \in \mathbb{N}$ *we have that the limit*

$$Px = \lim_{t\to\infty} \frac{\Gamma(r+m+1)}{\Gamma(m)} \frac{1}{t^{m+r}} \int_0^t (t-s)^{m-1} R(s)x\, ds$$

exists for all $x \in Ker(A) \oplus \overline{Ran(A)}$ *and is a bounded linear projection with*

$$\left\| \frac{\Gamma(r+m+1)}{\Gamma(m)} \frac{1}{t^{m+r}} \int_0^t (t-s)^{m-1} R(s)x\, ds - Px \right\| = O\left(\frac{1}{t^m}\right),$$

for all $x \in Ker(A) \oplus \overline{Ran(A)}$.

5. Application to k-convoluted semigroups

The notion of $K(t)$-convoluted semigroups and local-convoluted semigroups was introduced in 1995 by Cioranescu [6] and Cioranescu and Lumer [7]. The concept seems easier to handle than distribution and ultradistribution semigroups to which they turn out to be equivalent. In this section, by making use of our results, we are able to develop the corresponding ergodic theory.

In our terminology, a $k(t)$-convoluted semigroup corresponds to a $(k,1)$-regularized family $\{R(t)\}_{t\geqslant 0}$.

Our main results in this section can be stated as follows:

Theorem 6. *Let A be the generator of a k-convoluted semigroup $\{R(t)\}_{t\geqslant 0}$ satisfying*

$$||R(t)|| \leqslant Mk(t) \quad \text{for all } t \geqslant 0$$

and assume that $k(t)$ is positive and nondecreasing and such that (C) is satisfied. More precisely let

$$\lim_{t\to\infty} \frac{k(t)}{(1*k)(t)} = 0,$$

$$\sup_{t>0} \frac{tk(t)}{(1*k)(t)} < \infty,$$

$$\lim_{t\to\infty} \frac{(t*k)(t)}{(1*k)(t)} = \infty.$$

Then the same conclusions of Theorem 1 *are valid where*

$$Px := \lim_{t\to\infty} \frac{1}{(1*k)(t)} \int_0^t R(s)x\, ds$$

for all $x \in D(P) = Ker(A) \oplus \overline{Ran(A)}$,

$$B_1 y = -\lim_{t\to\infty} \frac{1}{(1*k)(t)} \int_0^t (t-s)R(s)y\, ds$$

for all $y \in D(B_1) = A(D(A) \cap \overline{Ran(A)})$. *Moreover, the rate of convergence of the limits has order* $\frac{k(t)}{(1*k)(t)}$.

Theorem 7 (Uniform Ergodic Theorem). *Let A be the generator of a k-convoluted semigroup $\{R(t)\}_{t\geqslant 0}$ satisfying*

$$||R(t)||\leqslant Mk(t) \quad \textit{for all } t\geqslant 0,$$

and assume (C). *Then assertions* (a) *and* (b) *of Theorem* 2 *are true, with* $a(t)=1$.

Regarding the abelian ergodic theorems for k-convoluted semigroups, we notice that there is no dependence on the function $k(t)$ for the ergodic nets $A_\lambda := \lambda(\lambda - A)^{-1}$ and $B_\lambda := -(\lambda - A)^{-1}$, $\lambda > 0$, by the general definition given in Section 3. Moreover, the rate of convergence depends only on the asymptotic behavior of $\hat{a}(\lambda) = \frac{1}{\lambda}$ as $\lambda \to \infty$. Thus, the corresponding abelian ergodic theorems for the generator A yield the same type of results as in the semigroup case.

6. Application to integral resolvents

We consider the integral equation

$$R(t)x = a(t)x + A\int_0^t a(t-s)R(s)x\,ds, \quad t\geqslant 0. \tag{14}$$

A solution of (14) is called integral resolvent for the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t\geqslant 0, \tag{15}$$

where $f\in C(\mathbb{R}_+; X)$; see [17] and the references therein.

Suppose $R(t)$ is an integral resolvent for (15). Then

$$u(t) = f(t) + A\int_0^t R(t-s)f(s)\,ds, \quad t\geqslant 0, \tag{16}$$

yields a mild solution to (15).

Integral resolvents have been studied extensively in the finite-dimensional case, where it is usually called the resolvent for (15); see, e.g. the monograph by Gripenberg et al. [9]. For equations with unbounded operators in infinite dimensions, the theory has been developed by many authors; see, e.g., [17, Theorem 1.4 and Section 10.6], and also the recent works [14,15]. Note that, in our terminology, an integral resolvent corresponds to an (a,a)-regularized family.

Following the same arguments as in the sections above, we can state ergodic theorems with rates for integral resolvents. The ergodic net is given by

$$A_t x := \frac{1}{(a*a)(t)}\int_0^t a(t-s)R(s)x\,ds; \quad x\in X, \ t>0 \tag{17}$$

with companion net

$$B_t x = \frac{-1}{(a*a)(t)}\int_0^t a(t-s)\int_0^s a(s-r)R(r)x\,dr\,ds; \quad x\in X, \ t>0.$$

Also, the estimator of the convergence rate of A_t is

$$e(t) = \frac{a(t)}{(a * a)(t)}.$$

Acknowledgments

The authors wish to thank the referee for many valuable suggestions in order to improve a previous version of this paper. The authors are supported in part by FONDECYT Grants #1010675 and #1000811.

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